Feb 1, 2023

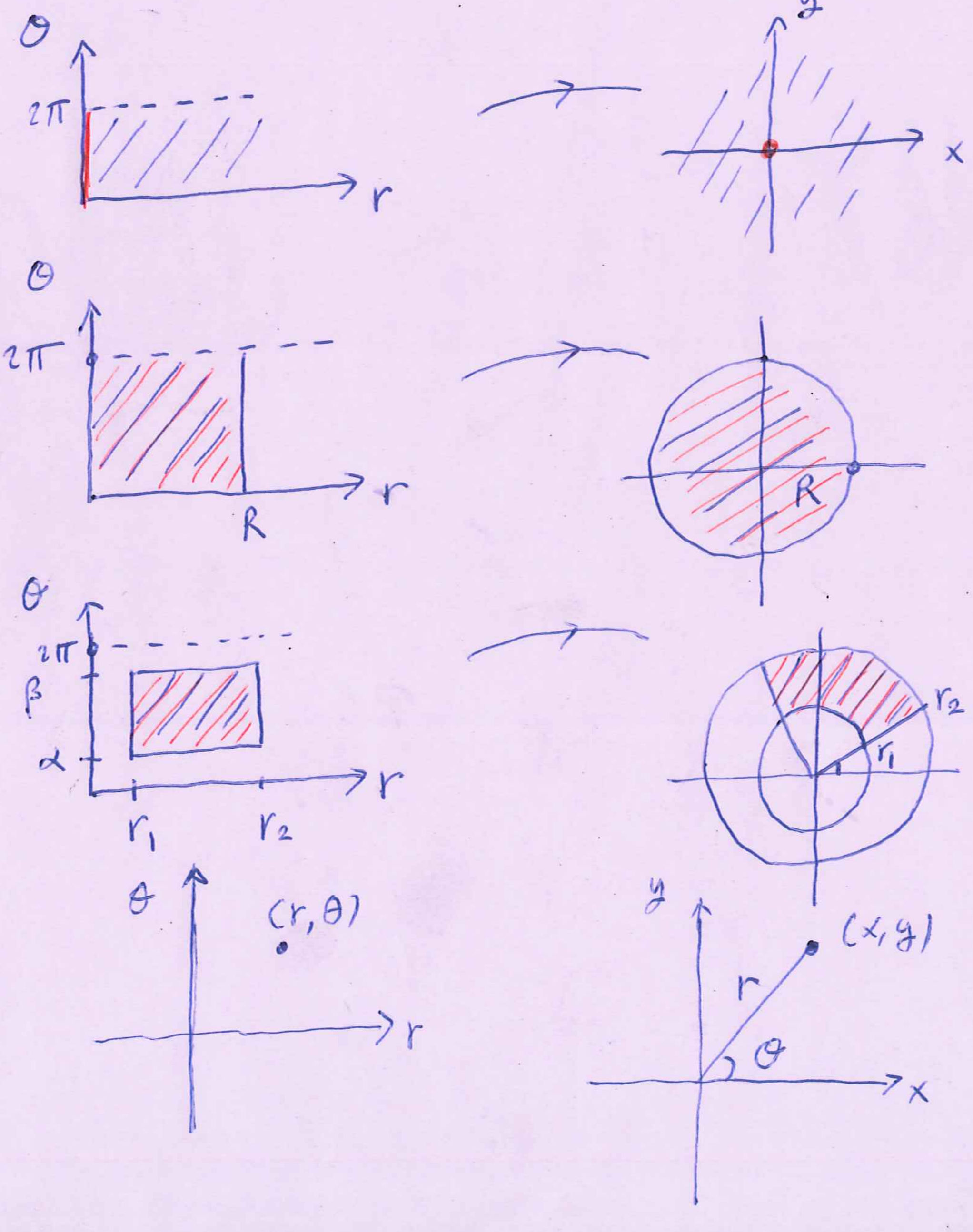
Week 4

2020B Adv. Cal. II

Any point (x, y) in the plane can be described in polar coordinates as (r, θ):

$$x = r\cos\theta\ , \quad y = r\sin\theta .$$

The correspondence $(r, \theta) \mapsto (x, y)$ sets up a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$. Restricted (r, θ) to $[0, \infty) \times [0, 2\pi]$, this mapping maps onto $\mathbb{R}^2$, and restricted to $(0, \infty) \times [0, 2\pi)$, it is 1-1 onto $\mathbb{R}^2$ except (omitting) the origin $(0, 0)$.



Let D be a fan-shaped region in (x,y)-plane

$$\alpha \leq \theta \leq \beta$$
$$R_1 \leq r \leq R_2$$

and $f(x,y)$ is a function on D. Then

$$\hat{f}(r,\theta) = f(r\cos\theta, r\sin\theta)$$

is a function on the rectangle $\hat{R} = [R_1, R_2] \times [\alpha, \beta]$.

<u>Theorem</u> f integrable on D. Then

$$\iint_D f(x,y)\, dA(x,y) = \iint_R \hat{f}(r,\theta)\, r\, dA(r,\theta)$$
$$= \int_\alpha^\beta \int_{R_1}^{R_2} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta .$$

Will discuss the proof later.

Now, suppose D is described by

$$\left\{ (x,y) : \begin{array}{l} \alpha \leq \theta \leq \beta,\ \rho_1(\theta) \leq r \leq \rho_2(\theta), \\ x = r\cos\theta,\ y = r\sin\theta \end{array} \right\}$$

then

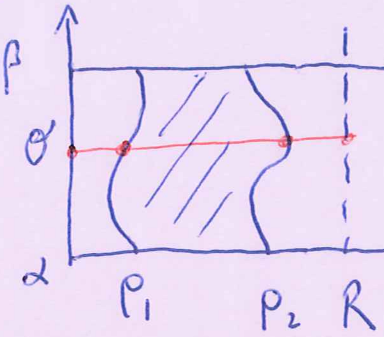
$$\boxed{\iint_D f(x,y)\, dA(x,y) = \int_\alpha^\beta \int_{\rho_1(\theta)}^{\rho_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta}$$

This is our basic formula.

$\square$ Pf: Let $\tilde{f}$ be the universal extension of f.

Let D_0 be the sector $\{(x,y) : \alpha \leq \theta \leq \beta, 0 \leq r \leq R\}$ where

R is so larger to make sure $D \subset D_0$. then



$$\iint_D f\,dA = \iint_{D_0} \tilde{f}\,dA$$

$$= \int_\alpha^\beta \int_0^R \hat{\tilde{f}}(r,\theta)\,r\,dr\,d\theta$$

For $\theta \in [\alpha, \beta]$, the horizontal line first hits $\rho_1(\theta)$, then $\rho_2(\theta)$, and finally R.

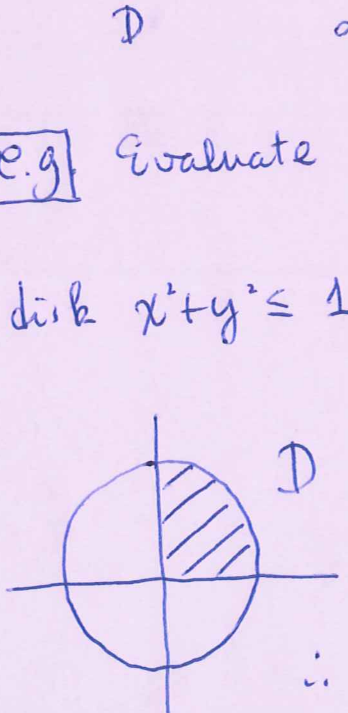
$$\int_0^R \hat{\tilde{f}}(r,\theta)\,r\,dr = \int_0^{\rho_1(\theta)} \hat{\tilde{f}}\,r + \int_{\rho_1(\theta)}^{\rho_2(\theta)} \hat{\tilde{f}}\,r + \int_{\rho_2(\theta)}^{R} \hat{\tilde{f}}\,r$$

$$= \int_{\rho_1(\theta)}^{\rho_2(\theta)} \hat{\tilde{f}}\,r\,dr \qquad \left(\because \hat{\tilde{f}} = 0 \text{ on } [0, \rho_1(\theta)] \text{ and } [\rho_2(\theta), R]\right)$$

$$= \int_{\rho_1(\theta)}^{\rho_2(\theta)} \hat{f}\,r\,dr \qquad \left(\because \hat{\tilde{f}} = \hat{f} \text{ on } [\rho_1(\theta), \rho_2(\theta)]\right)$$

$$\therefore \iint_D f\,dA = \int_\alpha^\beta \int_{\rho_1(\theta)}^{\rho_2(\theta)} \hat{f}(r,\theta)\,r\,dr\,d\theta. \quad \#$$

$\boxed{\text{e.g.}}$ Evaluate $\iint_D xy\,dA$ where D is the portion of the disk $x^2+y^2 \le 1$ in the first quadrant.



D is the sector

$0 \le \theta \le \pi/2$

$0 \le r \le 1$

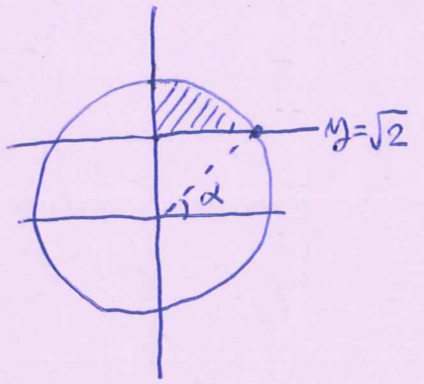
$$\therefore \iint_D xy\,dA = \int_0^{\pi/2} \int_0^1 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 r^3 \cos\theta \sin\theta \, dr \, d\theta$$

$$= \frac{1}{8} \int_0^{\frac{\pi}{2}} \sin 2\theta \, d\theta$$

$$= \frac{1}{8} . \#$$

e.g. Let D be the region bounded by $x^2+y^2=4$, $y=\sqrt{2}$ and the y-axis. Find its area.



The line $y=\sqrt{2}$ intersects $x^2+y^2=4$ at $(\sqrt{2},\sqrt{2})$. D is expressed as

$$\alpha \le \theta \le \pi/2$$

$$\frac{\sqrt{2}}{\sin\theta} \le r \le 2$$

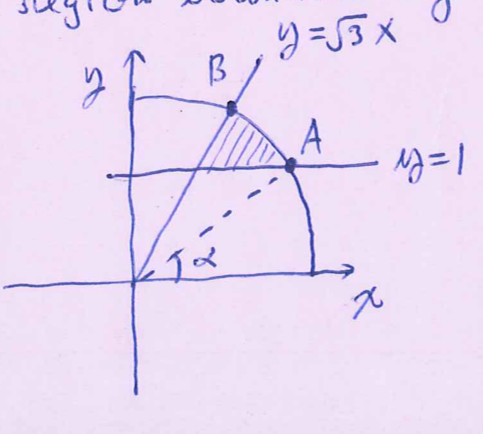
Here $\tan\alpha = \sqrt{2}/\sqrt{2} = 1 \Rightarrow \alpha = \pi/4$.

$$\therefore \text{area} = \int_{\pi/4}^{\pi/2} \int_{\frac{\sqrt{2}}{\sin\theta}}^{2} r \, dr \, d\theta = \int_{\pi/4}^{\pi/2} \left. \frac{r^2}{2} \right|_{\frac{\sqrt{2}}{\sin\theta}}^{2} d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left(2 - \frac{1}{\sin^2\theta}\right) d\theta = 2\left(\pi/2 - \pi/4\right) - \int_{\pi/4}^{\pi/2} \csc^2\theta \, d\theta$$

$$= \frac{\pi}{2} - \cot\theta \Big|_{\pi/4}^{\pi/2} = \frac{\pi}{2} - 1 . \#$$

e.g. Let D be the region bounded by $x^2+y^2=4$, $y=1$, $y=\sqrt{3}x$. Find its area.



$A(\sqrt{3}, 1)$

$B(1, \sqrt{3})$

$$\tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \pi/6$$

$$\tan\beta = \frac{\sqrt{3}}{1} \Rightarrow \beta = \pi/3$$

$\therefore$ D : $\pi/6 \le \theta \le \pi/3$

$$1/\sin\theta \le r \le 2$$

$$\text{area} = \int_{\pi/6}^{\pi/3} \int_{1/\sin\theta}^{2} r \, dr \, d\theta = \cdots = \frac{\pi - \sqrt{3}}{3}.$$

e.g. Convert

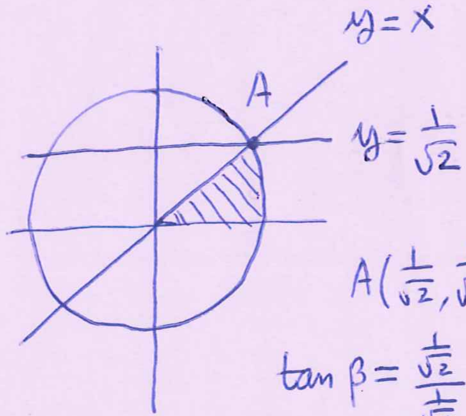
$$\int_0^{\frac{1}{\sqrt{2}}} \int_y^{\sqrt{1-y^2}} \sqrt{x^2+y^2} \, dx \, dy$$

into polar coordinates and then evaluate it.

D is in rectangular coordinates :

$$0 \le y \le \frac{1}{\sqrt{2}}$$

$$y \le x \le \sqrt{1-y^2}$$



$A\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$\tan\beta = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1$$

$\therefore \beta = \pi/4$

In polar coordinates,

$$0 \le \theta \le \pi/4$$

$$0 \le r \le 1$$

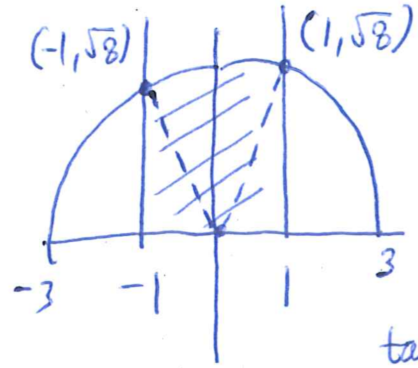
$\therefore$ The integral

$$= \int_0^{\pi/4} \int_0^1 r \, r \, dr \, d\theta$$

$$= \frac{1}{3} \frac{\pi}{4} = \frac{\pi}{12} \quad \#$$

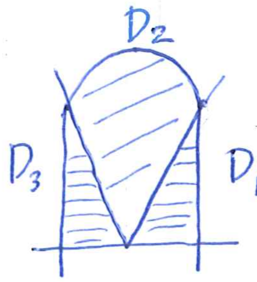
e.g "Compound regions"

Let D be the region bounded above by $x^2+y^2=9$, over the interval $[-1,1]$ in the x-axis.



Express
$$\iint_D f\,dA$$
in polar coordinates.

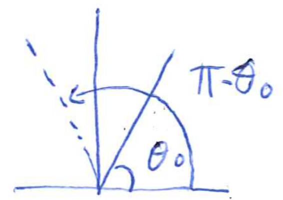
$\tan\theta_0 = \frac{\sqrt{8}}{1}$



D_1 : $0 \le \theta \le \theta_0$, $\theta_0 = \tan^{-1}\sqrt{8}$,
$0 \le r \le \frac{1}{\cos\theta}$,

D_2 : $\theta_0 \le \theta \le \pi - \theta_0$
$0 \le r \le 3$

$D = D_1 \cup D_2 \cup D_3$



D_3 : $\pi - \theta_0 \le \theta \le \pi$
$0 \le r \le \frac{-1}{\cos\theta}$,

$$\iint_D f\,dA = \int_0^{\theta_0}\int_0^{\frac{1}{\cos\theta}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$
$$+ \int_{\theta_0}^{\pi-\theta_0}\int_0^{3} f(\ldots)\,r\,dr\,d\theta + \int_{\pi-\theta_0}^{\pi}\int_0^{\frac{-1}{\cos\theta}} f(\ldots)\,r\,dr\,d\theta.$$

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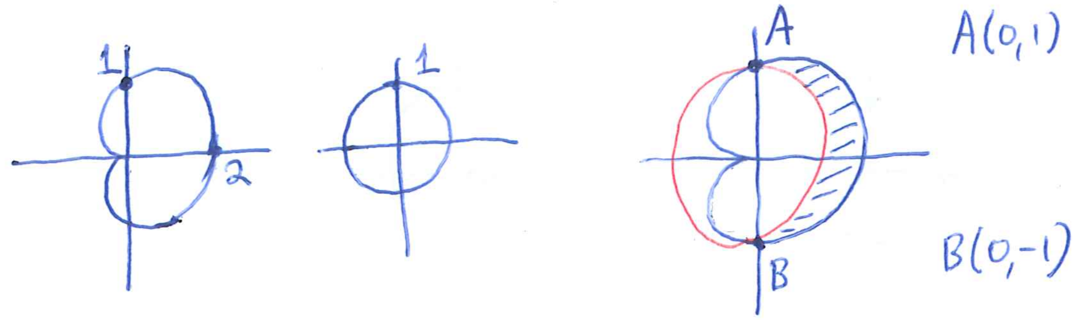
e.g Find the area of the region lying inside the cardioid $r = 1+\cos\theta$ and outside the circle $r=1$.



$D: -\pi/2 \le \theta \le \pi/2$

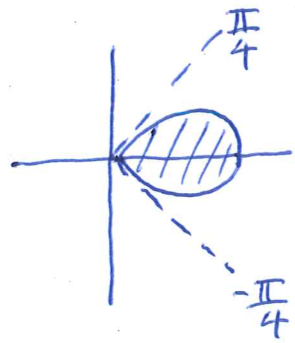
$1 \le r \le 1 + \cos\theta$.

$$\text{area} = \int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos\theta} r\,dr\,d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (1+\cos\theta)^2 - 1 \, d\theta$$

$\vdots$

$= 2 + \pi/4$.

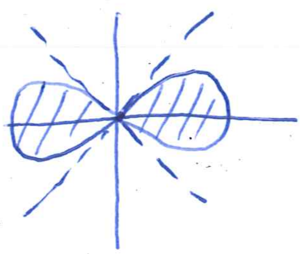
$\boxed{\text{eg}}$ Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$. $\cos 2\theta$ is π-periodic. We first sketch its graph over $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



$\cos 2\theta \geq 0$ for $\theta \in [-\pi/4, \frac{\pi}{4}]$

$< 0 \quad \theta \in (\frac{\pi}{4}, \frac{\pi}{2}), (-\pi/2, -\pi/4)$

By π-periodicity, it has 2 leaves.



$$\text{area} = 2 \int_{-\pi/4}^{\pi/4} \int_{0}^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta$$

$\vdots$

$= 4$ #

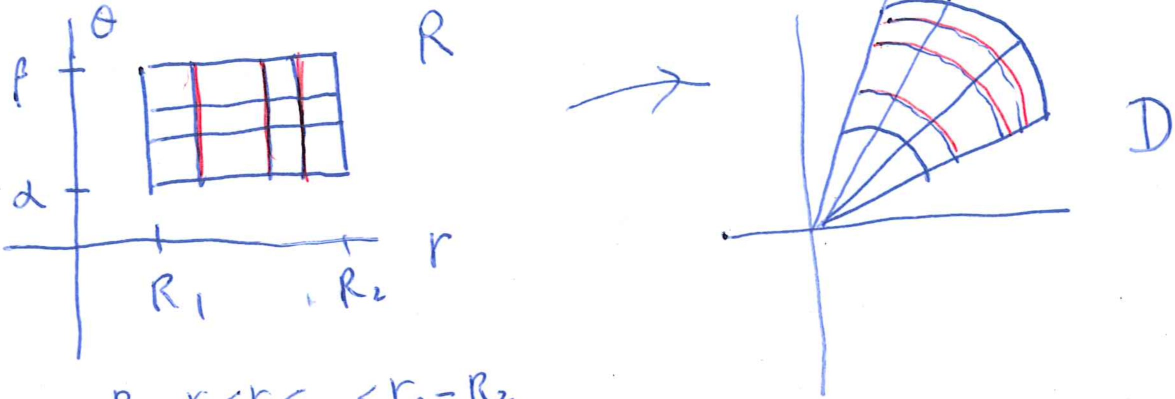
Now, we sketch a proof of the theorem on Pg 2:

$$\iint_D f\,dA = \int_\alpha^\beta \int_{R_1}^{R_2} \hat{f}(r,\theta)\, r\, dr\, d\theta \text{ , where}$$

D is the fan-shaped region

$$\{(x,y) : x = r\cos\theta,\ y = r\sin\theta,\ \alpha \le \theta \le \beta,\ R_1 \le r \le R_2\}$$

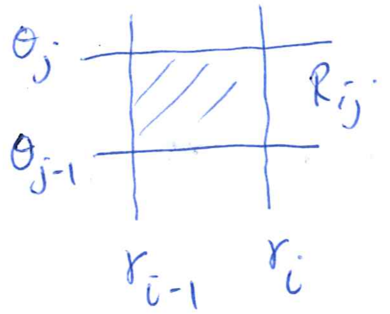
Consider the partition P on $R = [\alpha,\beta]\times[R_1,R_2]$

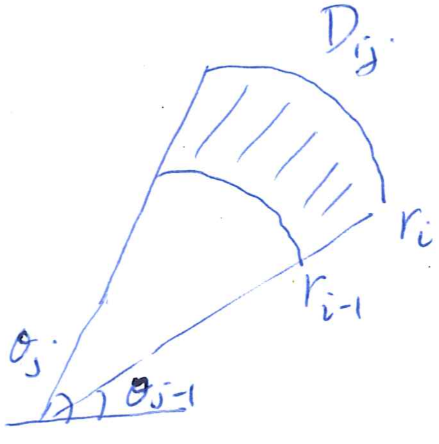


$R_1 = r_0 < r_1 < \dots < r_n = R_2$

$\alpha = \theta_0 < \theta_1 < \dots < \theta_m = \beta$

R_{ij} go over to D_{ij}





Now

$$\iint_D f\,dA = \sum_{i,j} \iint_{D_{ij}} f\,dA \approx \sum_{i,j} \iint_{D_{ij}} f(P_{ij})\,dA$$

When D_{ij} v. small, f is almost constant on D_{ij}, $f(x,y) \approx f(P_{ij})$, P_{ij} = midpoint of D_{ij}.

$$= \sum_{i,j} f(P_{ij}) \iint_{D_{ij}} dA$$

$$= \sum_{i,j} f(P_{ij}) \text{ area of } D_{ij}.$$

area of $D_{ij} = \frac{1}{2} r_i^2 \Delta\theta_j - \frac{1}{2} r_{i-1}^2 \Delta\theta_j$

$= \frac{1}{2}(r_i + r_{i-1}) \Delta r_i \Delta\theta_j$

$= \bar{r}_i \Delta r_i \Delta\theta_j, \quad \bar{r}_i = \frac{1}{2}(r_i + r_{i-1})$

$$\therefore \iint_D f\,dA \approx \sum_{i,j} f(P_{ij}) \bar{r}_i \Delta r_i \Delta\theta_j \quad \text{— ①}$$

On the other hand,

$$\int_\alpha^\beta \int_{R_1}^{R_2} \hat{f}(r,\theta)\, r\,dr\,d\theta = \iint_R \hat{f}(r,\theta)\, r\, dA(r,\theta)$$

taking the tag point to be the midpoint

$$q_{ij} = (\bar{r}_i, \bar{\theta}_j), \quad \bar{\theta}_j = \frac{1}{2}(\theta_{j-1} + \theta_j),$$

the above integral

$$\approx \sum_{i,j} \hat{f}(\bar{r}_i, \bar{\theta}_j) \bar{r}_i \Delta r_i \Delta\theta_j$$

$$= \sum_{i,j} f(P_{ij}) \bar{r}_i \Delta r_i \Delta\theta_j \quad \text{— ②}$$

① = ②, $\therefore$ ~~As~~ $\|P\| \to 0$,

$$\int_\alpha^\beta \int_{R_1}^{R_2} \hat{f}(r,\theta)\, r\,dr\,d\theta = \iint_D f\,dA \quad . \ \#$$